NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

A X1 **Indices and Logarithms**

With references to

Name: ....

Initial version by H. Lam, September 2014 (Graphs of exponential/logarithmic functions). Last updated August 23, 2022. Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

(!) Beware! Heed warning.

(A) Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

(L) Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-6 manipulates and solves expressions using the logarithmic and index laws, and uses logarithms and exponential functions to solve practical problems

Syllabus subtopics

MA-E1 (1.1, 1.2, 1.4) Logarithms and Exponentials

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Mathematics Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) or *CambridgeMATHS Mathematics Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Ⓡ Review of index laws

Learning Goal(s)

Knowledge
What are the index laws

Skills
Applying the index laws to simplify expressions

Understanding
How to derive the index laws

By the end of this section am I able to:
6.1 Use the index laws

1.1 Operations resulting in addition/subtraction/multiplication of indices

Only the most challenging questions are provided here. The rest are omitted for brevity and should be assumed knowledge.

Laws/Results

For $a > 1$:

(Product) $$a^x \times a^y = a^{x+y} \tag{1.1}$$

(Quotient) $$a^x \div a^y = a^{x-y} \tag{1.2}$$

(Power) $$(a^x)^y = a^{x-y} \tag{1.3}$$

(Power 0) $$a^0 = 1 \tag{1.4}$$

(Negative index) $$a^{-x} = \frac{1}{a^x} \tag{1.5}$$

(Fractional index) $$a^{\frac{1}{x}} = \sqrt[x]{a} \tag{1.6}$$

Laws/Results

(A slightly more neglected rule)

$$(ab)^x = a^x b^x \tag{1.7}$$

Important note

Use these laws in either "direction", i.e. from L to R *or* R to L

Example 1

[Ex 8A Q15(e)] Write as a single fraction, without negative indices and simplify:

$$x^{-2}y^{-2}\left(x^2y^{-1} - y^2x^{-1}\right)$$

Important note

An everyday, plain English definition of *simplify*: write the expression with ....less........ ....strokes........ .

Example 2

Fully simplify: $\dfrac{2^{n+1} - 2^n}{2^{2n+1} - 2^{2n}}$

Answer: 2^{-n}

Example 3

[Ex 8A Q16(f)] Fully simplify:

$$\frac{24^{x+1} \times 8^{-1}}{6^{2x}}$$

Answer: $2^x 3^{1-x}$

Example 4

[Ex 8A Q18(c)] Fully simplify: $\dfrac{12^n - 18^n}{3^n - 2^n}$.

Answer: -6^n

1.2 Fractional indices

Example 5

Fully simplify: $\sqrt[3]{\dfrac{x^{5k+1}y^{4k-5}}{x^{2(k-1)}y^{-2k+1}}}$

Answer: $x^{k+1}y^{2(k-1)}$

Further exercises

Ex 8A

- Q5-19

Ex 8B

- Q5-19, 21

Section 2

Ⓡ Logarithms

Learning Goal(s)

Knowledge
What are the logarithmic laws

Skills
Applying the logarithm laws to simplify expressions

Understanding
How to derive the logarithmic laws

By the end of this section am I able to:

6.2 Define logarithms as indices: $y = a^x$ is equivalent to $x = \log_a y$, and explain why this definition only makes sense when $a > 0$ and $a \neq 1$

6.3 Understand how to use a calculator to find logarithms base 10.

6.4 Recognise and use the inverse relationship between logarithms and exponentials.

6.5 Derive the logarithmic laws from the index laws and use the algebraic properties of logarithms to simplify and evaluate logarithmic expressions. exponentials.

2.1 Basic rules

Definition 1

If $a^x = y$, then $x = \log_a y$.

2.2 Laws of logarithms

Laws/Results

$$\text{(Product)} \qquad \log x + \log y = \log(\quad) \tag{2.1}$$

$$\text{(Quotient)} \qquad \log x - \log y = \log\left(\quad\right) \tag{2.2}$$

$$\text{(Power)} \qquad \log(x^a) = \underline{a \log x} \tag{2.3}$$

$$\text{(Power 1)} \qquad \log_a a = \underline{1} \tag{2.4}$$

$$\text{(Power 0)} \qquad \log_a 1 = \underline{0} \tag{2.5}$$

Important note

Use these laws in either "direction", i.e. from L to R *or* R to L

Example 6

Given $\log_a 2 = m$ and $\log_a 3 = n$, write the following in terms of m and n:

(a) $\log_a 4$ (c) $\log_a 1.5$ (e) $\log_a 18$

(b) $\log_a 27$ (d) $\log_a 12$ (f) $\log_a 13.5$

Example 7

[**1995 2U HSC Q7**] Given $\log_a b = 2.75$ and $\log_a c = 0.25$, find the value of

i $\log_a \left(\dfrac{b}{c}\right)$ **1**

ii $\log_a (bc)^2$ **2**

Example 8

Write each of the following as $\log_2 3$:

(a) $\log_2 81$ (b) $\log_2 2\sqrt{3}$ (c) $\log_2 \frac{8}{9}$

2.3 Exponential analog & mutual inverse

Laws/Results

(Mutual inverse)

$$a^{\log_a x} = x \qquad \log_a a^x = x \tag{2.6}$$

Example 9

Express 5 as a power of 2.

Example 10

[**Ex 8D Q15**] Fully simplify: $2^{\log_2 3 + \log_2 5}$

Further exercises

Ex 8C (Pender et al., 2019b)

- Q5-11

Ex 8D (Pender et al., 2019b)

- Q5-16

Ex 12A (Pender, Sadler, Shea, & Ward, 1999)

- Q3-16, Ⓔ Q17-18

Section 3

Equations involving indices and logarithms

Learning Goal(s)

Knowledge
What is the change of base law

Skills
Applying the logarithm laws to solve equations

Understanding
How to make a pronumeral the subject in equations involving exponentials and logarithms

By the end of this section am I able to:

6.7 Consider different number bases and prove and use the change of base law $\log_a x = \dfrac{\log_b x}{\log_b a}$

6.8 Can solve a range of equations involving indices and logarithms

3.1 Change of base

Steps

To change from base a for $\log_a x$ to base b:

1. Let $y = \log_a x$.

2. Rewrite as exponential:

$$a^y = x$$

3. Take logarithm to base b:

$$\log_b a^y = \log_b x$$

4. Use log-power law (2.3):

$$y \log_b a = \log_b x$$

5. F Divide and rewrite:

$$y = \frac{\log_b x}{\log_b a}$$

3.1.1 Examples

Example 11

Solve $2^x = 7$ correct to 4 significant figures.

Answer: 2.807

Example 12

[2010 NSB Ext 1 Prelim Yearly]

i Rewrite $2^x = 5^y$ with $\frac{x}{y}$ as the subject by taking logarithms to a suitable base. **1**

ii Hence or otherwise, find the exact value of $8^{\frac{x}{y}}$ if $2^x = 5^y$. **2**

Answer: (i) $\log_2 5$ (ii) 125

3.2 Harder equations

3.2.1 Exponential

Example 13

Solve for x:

$$3^{2x} - 12 \times 3^x + 27 = 0$$

Answer: 1, 2

Example 14

Solve for x:

$$4^x - 12 \times 2^x + 32 = 0$$

Answer: 2, 3

3.2.2 Logarithmic

Example 15

Solve: $2\log_3 x = \log_3(6-x)$.

Answer: 2

Example 16

Solve: $\log 16 - \log x = \log(8-x)$

Example 17

[2019 NBHS Ext 1 Task 2] Find the value of x if

$$\log_a 2 + 2\log_a x - \log_a 6 = \log_a 3$$

Answer: 3

Example 18

Solve the following pair of simultaneous equations:

$$\begin{cases} 5^{x+y} = \frac{1}{5} \\ 5^{3x+2y} = 1 \end{cases}$$

Answer: $x = 2$, $y = -3$

Example 19

(Pender et al., 1999, Ex 12A)

(a) Use the change of base rule to show that

$$\log_{a^x} b = \frac{\log_a b}{x}$$

(b) Hence evaluate $\log_{\sqrt{27}} 81$ without a calculator.

(c) Solve for x:

$$\log_{\sqrt{a}}(x+2) - \log_{\sqrt{a}} 2 = \log_a x + \log_a 2$$

Answer: (b) $\frac{8}{3}$ (c) $x = 2$

Example 20

[**Ex 12A Q18**] (Pender et al., 1999) ! Solve for x: $\log_{2x} 216 = x$

Answer: 3

Important note

Hint: Rewrite in index form, then consider the prime factorisation of 216.

Further exercises

Ex 8B

- Q20

Ex 8E

- Q2, 4-15, E Q16

Section 4

Exponential and logarithmic graphs

4.1 Exponential

Learning Goal(s)

Knowledge
What are the important features of exponential and logarithmic graphs

Skills
Sketch the graphs of exponentials and logarithms

Understanding
The techniques used to determine the important features on exponential and logarithmic graphs

By the end of this section am I able to:

6.10 Recognise and sketch the graphs of $y = ka^x$, $y = ka^{-x}$ where k is a constant, and $x = \log_a y$

6.11 Solve a range of problems related to exponential and logarithmic functions

Theorem 1

Ⓜ Basic exponential curves:

- Equation: $y = a^x$
- Condition on a: $a > 0$
- Domain: all real x
- Range: $y > 0$
- When $x = 0$, $y = 1$
- When $x = 1$, $y = a$
- As
  - $\lim\limits_{x \to \infty} a^x = \infty$.
  - $\lim\limits_{x \to -\infty} a^x = 0$.
- Inverse: $y = \log_a x$

Sketch:

- Always show two points on the curve!

4.1.1 Transformations

Example 21

Sketch:

1. $y = 3^x$
2. $y = 3^{-x}$
3. $y = 10^x$
4. $y = \left(\dfrac{1}{6}\right)^x$
5. $y = 2^{x-1}$
6. $y = 2^x - 1$
7. $y = 1 - 2^{-x}$

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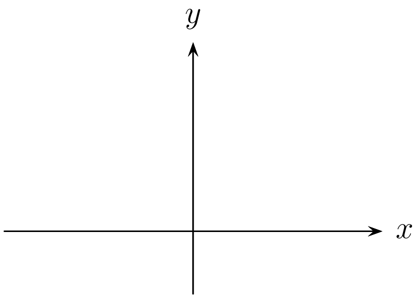

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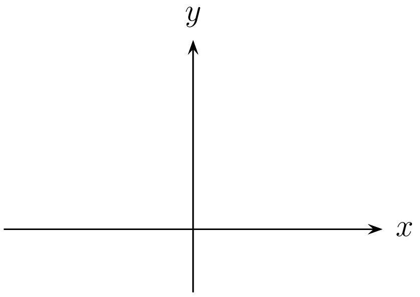

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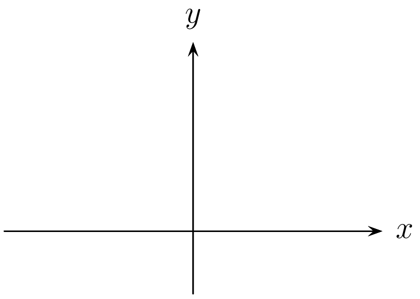

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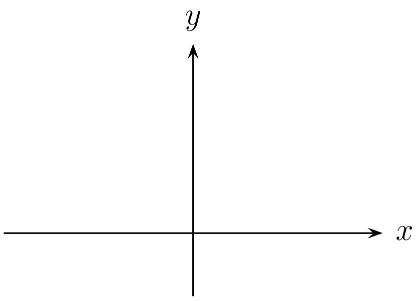

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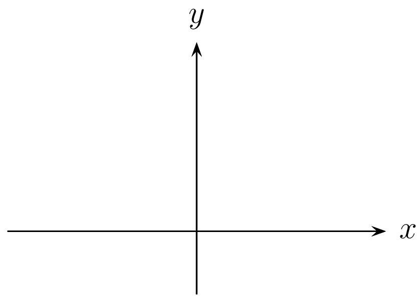

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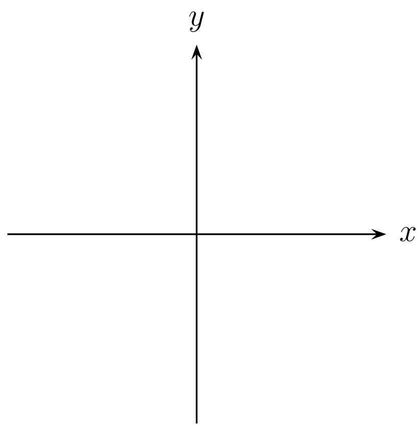

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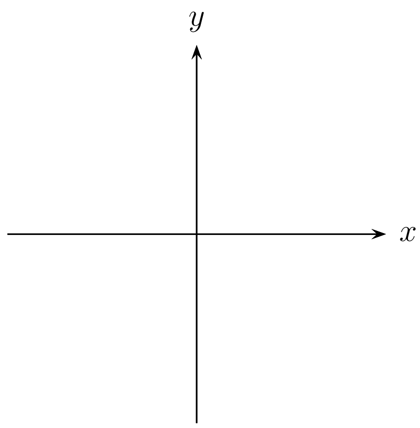

4.1.2 (X1) Harder graphs

Example 22

Sketch $y = 2^{|x|}$

Example 23

Sketch $y = 2^x + 2^{-x}$

4.2 Logarithmic

Theorem 2

Basic logarithmic curves:

- Equation: $y = \log_a x$
- Condition on a: $a > 0$
- Domain: $x > 0$
- Range: all real y
- When $x = 1$, $y = 0$
- When $x = a$, $y = 1$
- As
  - $x \to \infty$, $y \to \infty$
  - $x \to 0^+$, $y \to$ 000
- Inverse: $y = a^x$

Sketch:

- Always show two points on the curve!

4.2.1 Transformations

Example 24

1. $f(x) = \log_{10}(2x)$.
2. $f(x) = \log_5 x^2$, $x > 0$.
3. $f(x) = \log_6(3x - 6)$.
4. $f(x) = \log_7(2x + 3)$.
5. $f(x) = \log_{10}|x|$.
6. $f(x) = \log_7(-x)$.

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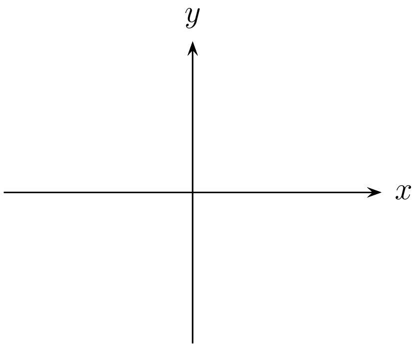

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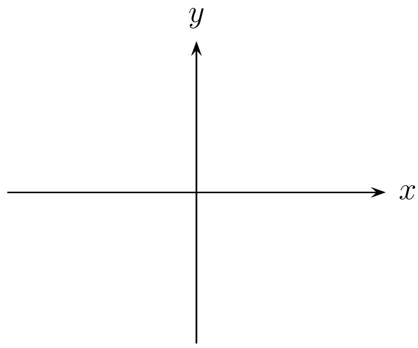

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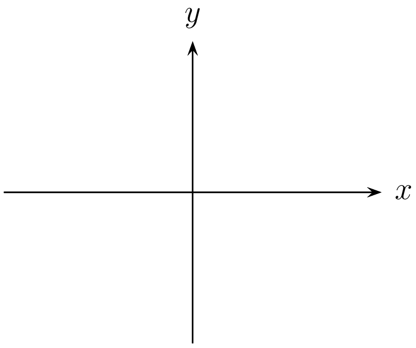

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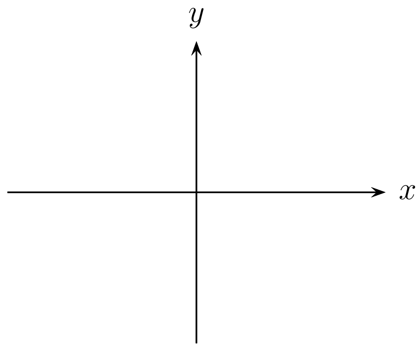

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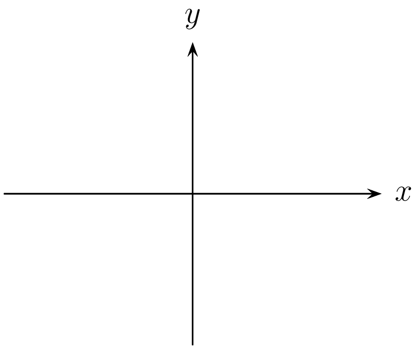

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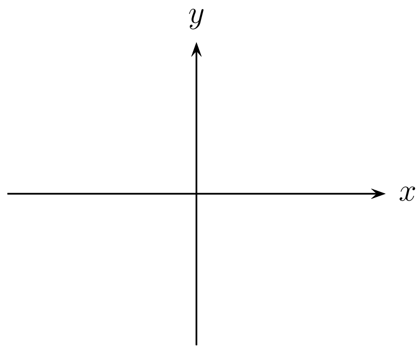

Further exercises

Ex 8F

- Q2-8

4.2.2 (X1) Harder graphs

Example 25

Sketch $y = \log_5 \sqrt{x}$

Example 26

Sketch $y = \log_3 |x|$

Example 27

Sketch $|y| = \log_{10}(-x)$

Example 28

Sketch $y^2 = \log_{10}(-x)$

Further exercises

Ex 8F

- Q9-11

Section 5

Applications

Learning Goal(s)

Knowledge
Solving real-life problems using exponentials and logarithms

Skills
Constructing equations involving exponentials and logarithms to solve from real life contexts

Understanding
The real life applications of exponentials and logarithms

By the end of this section am I able to:

6.6 Interpret and use logarithmic scales, for example decibels in acoustics, different seismic scales for earthquake magnitude, octaves in music or pH in chemistry

6.9 Solve algebraic, graphical and numerical problems involving logarithms in a variety of practical and abstract contexts, including applications from financial, scientific, medical and industrial contexts

Important note

Most questions in this section make use of the exponent e instead of 2, 3 or 10. Simply treat e as another one of these numerals for the time being - there will be a more in depth study of e later on with calculus.

Example 29

[2019 CSSA 2U Trial Q16] (2 marks) Find the domain of the function

$$g(x) = \ln\left(x^2 - x\right)$$

Example 30

[2000 2U HSC Q9] with slight modifications

i. Sketch $y = \log_e x$ (Note: $e \approx 2.71828\cdots$) **2**

ii. On the same sketch, find, graphically, the number of solutions of the equation **1**

$$\log_e x - x = -2$$

Example 31

[1990 2U HSC Q7] A quantity Q of radium at time t in years is given by

$$Q = Q_0 e^{-kt}$$

where k is a constant and Q_0 is the initial amount of radium at time $t = 0$.

i. Given that $Q = \frac{1}{2}Q_0$, when $t = 1\,690$ years, calculate the value of k **1**

ii. After how many years does only 20% of the initial amount of radium remain? **2**

Example 32

[2020 CSSA Adv Trial Q18] A cup of coffee is cooling according to the following exponential formula

$$C = 21 + \left(74 \times 3^{-0.2t}\right)$$

where C is the temperature in degrees Celsius and t is the time in minutes since the coffee was poured.

(a) Calculate the initial temperature of the coffee. **1**

(b) Calculate the temperature of the coffee after 10 minutes, correct to the nearest degree. **1**

(c) After how many minutes, to the nearest minute, will the coffee first reach 50°C? **2**

Answer: (a) 95°C (b) 29.2°C (c) $\approx$ 4 min 16 seconds

Example 33

[1996 2U HSC Q3] A layer of plastic cuts out 15% of the light and lets through the remaining 85%.

i. Show that two layers of the plastic let through 72.25% of the light **1**

ii. How many layers of the plastic are required to cut out at least 90% of the light? **2**

Example 34

[1985 2U HSC Q10] A population $N(t)$ varies with time according to the law

$$N(t) = Ce^{kt}$$

where C, k are positive constants and $t \geq 0$

i. Show that, if a, b are two positive numbers such that $a + b = 1$, then **1**

$$N(at + bu) = \Big(N(t)\Big)^a \Big(N(u)\Big)^b$$

for any $t \geq 0$, $y \geq 0$.

ii. Hence or otherwise, find $N(13)$, given that $N(3) = 10$ and $N(18) = 100$ **2**

Example 35

[2011 2U HSC Q10] The intensity I, measured in watt/m^2, of a sound is given by

$$I = 10^{-12} \times e^{0.1L}$$

where L is the loudness of the sound in decibels.

i If the loudness of a sound at a concert is 110 decibels, find the intensity of the sound. Give your answer in scientific notation. **1**

ii Ear damage occurs if the intensity of a sound is greater than 8.1×10^{-9} watt/m^2. **2**

What is the maximum loudness of a sound so that no ear damage occurs?

iii By how much will the loudness of a sound have increased if its intensity has doubled? **2**

Example 36

[1990 2U HSC Q8] It is assumed that the number $N(t)$ of termites in a certain mound at time $t \geq 0$ is given by

$$N(t) = \frac{A}{2 + e^{-t}}$$

where $e \approx 2.7818\cdots$, A is a constant and t is measured in months.

i. At time $t = 0$, $N(t)$ is estimated at 3×10^5 termites. What is the value of A? **1**

ii. What is the value of $N(t)$ after one month? **1**

iii. How many termites would you expect to find in the mound when t is very large? **1**

iv. (X1 - added and modified) By considering the graph of $y = 2 + e^{-t}$, sketch the graph of $N(t)$.

Further exercises

Ex 8G

- All questions

5.1 Additional questions

1. [**1992 2U HSC Q4**] Ten kilograms of sugar is placed in a container of water and begins to dissolve. After t hours the amount A kg of undissolved sugar is given by

$$A = 10e^{-kt}$$

(where $e \approx 2.7818\cdots$)

i. Calculate k, given that $A = 3.2$ when $t = 4$. **1**

ii. After how many hours does 1 kg of sugar remain undissolved? **1**

2. [**2010 2U HSC Q4**] (2 marks) Let $f(x) = 1 + e^x$

Show that $f(x) \times f(-x) = f(x) + f(-x)$.

Answers

1. i. $k = \frac{1}{4}\ln\frac{10}{3.2} \approx 0.28$ ii. $t \approx 8.08$ hrs

References

Pender, W., Sadler, D., Shea, J., & Ward, D. (1999). *Cambridge Mathematics 3 Unit Year 11* (1st ed.). Cambridge University Press.

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